MTH311 Real Analysis I

EXAM Q1-3

1. (a) Clearly $a + S$ is also bounded and nonempty. Hence, by the Axiom of Completeness, both $\sup S$ and $\sup(a + S)$ exist.

   $\sup(a + S) \leq a + \sup S$.

   Let $y \in a + S$. Then $y = a + x$ for some $x \in S$. Then $x \leq \sup S$. Therefore $a + S \leq a + \sup S$. This is true for all $y \in a + S$. Therefore, $a + \sup S$ is an upper bound of $a + S$. Hence $\sup(a + S) \leq a + \sup S$.

   $a + \sup S \leq \sup(a + S)$. [We show that $\sup(a + S) - a$ is an upper bound of S.]

   Let $x \in S$. Then $a + x \in (a + S)$. Hence $a + x \leq \sup(a + S)$. Therefore, $x \leq \sup(a + s) - a$. This is true for all $x \in S$. Hence $\sup(a + S) - a$ is an upper bound of S.

   (b) We show that $-\inf S$ is an upper bound of $\sup(-S)$ and then show that it is the least among the upper bounds.

   Let $y \in -S$. Then $-y \in S$. Therefore, $-y \geq \inf S$, which implies that $y \leq -\inf S$. This shows that $-\inf S$ is an upper bound.

   Suppose s is an upper bound of $-S$. Then for all $x \in S$, we have $-x \in (-S)$. This means that $-x \leq s$, which implies that $x \geq (-s)$. Therefore, $-s$ is a lower bound of S. Hence $\inf S \geq (-s)$, in other words, $-\inf S \leq s$.

2. (a) Let $\epsilon > 0$ be given. Then there exists $N_1 \in \mathbf{N}$ such that $\frac{1}{N} < \frac{\epsilon}{3}$. Also, there exists $N_2 \in \mathbf{N}$ such that for all $m \geq n \geq N_2$,

   $$|b_m - b_n| < \frac{\epsilon}{3}.$$

   Take $N := \max\{N_1, N_2\}$. Then for all $m \geq n \geq N$,

   $$|a_m - a_n| \leq |a_m - b_m| + |b_m - b_n| + |b_n - a_n| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

   (b) (i) Since $\sum_{n=0}^{\infty} a_n$ converges, we know that $\lim_{n\to\infty} a_n = 0$. Therefore, there exists some $N \in \mathbf{N}$ such that for all $n \geq N$, we have $|a_n| \leq 1$. Then we see that for all $n \geq N$, we have $|a_n b_n| \leq |b_n|$. Clearly $\sum_{n=N}^{\infty} |b_n|$ converges. Therefore, by the Comparison Test, we find that $\sum_{n=N}^{\infty} |a_n b_n|$ converges. Therefore, $\sum_{n=0}^{\infty} |a_n b_n|$ converges.

   (ii) Take $(a_n) = \frac{(-1)^{n+1}}{n+1}$. Then it is clear that $\sum_{n=0}^{\infty} a_n$ which is the alternating harmonic series, converges. Let $b_0 = 1$, and for all $n \in \mathbf{N}^+$, let $b_n = 0$. Then clearly $\sum_{n=0}^{\infty} |b_n| = 1$ converges. However,

   $$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} |a_i b_j| = \sum_{i=0}^{\infty} \sum_{j=0}^{0} |a_i b_j| = \sum_{i=0}^{\infty} |a_i| = \sum_{i=0}^{\infty} \frac{1}{n+1}$$

   diverges.

3. Let $x \in f^{-1}(B)$. Then $f(x) \in B \subseteq f(A)$. Since B is open, there exists some $\epsilon > 0$ such that

$$V_\epsilon(f(x)) \subseteq B.$$

Since f is continuous, there exists some $\delta > 0$ such that

$$f(V_\delta(x)) \subseteq V_\epsilon(f(x)) \subseteq B.$$

Since f maps the whole set $V_\delta(x)$ into B, this means that

$$V_\delta(x) \subseteq f^{-1}(B).$$

FINAL EXAMINATION SOLUTIONS, MAS311 REAL ANALYSIS I, 2011/12

QUESTION 1. **(X marks)**

(i)

(ii)

QUESTION 2. **(X marks)**

(i)

(ii)

QUESTION 3. **(20 marks)**

(i) Prove that for all $x > 0$ and $n \in \mathbb{N}$, there is $t \in (0, x)$ such that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \frac{e^t}{(n+1)!}x^{n+1}.$$

Proof. If $f(x) = e^x$, then $f^{(n)}(x) = e^x$ for all $n = 0, 1, 2, \cdots$. So by a direct application of Taylor's theorem, we have the above identity. ☐

(ii) Show that for all $n \in \mathbb{N}$, there is R_n such that

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + R_n \text{ and } 0 < R_n < \frac{3}{(n+1)!}.$$

Proof. Using the result of part (i), we have

$$e^x = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \frac{e}{(n+1)!}.$$

We may take

$$R_n = \frac{e}{(n+1)!}.$$

Since $0 < e < 3$, we have the desired result. ☐

(iii) Prove that e must be irrational.
Hint: If $e = a/b$ with $a, b \in \mathbb{N}$, then take $n \in \mathbb{N}$ with $n > \max\{b, 3\}$ and compute $n!e$ using the result of (ii). Is $n!R_n$ an integer? How large and how small can it be?

Proof. If $e = a/b$ with $a, b \in \mathbb{N}$, then take $n \in \mathbb{N}$ with $n > \max\{b, 3\}$. The result of (ii) gives that

$$n!R_n = n!\frac{a}{b} - n!\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right).$$

The right-hand side of the above is an integer. Therefore, $n!R_n \in \mathbb{Z}$. But the bounds on R_n from part (ii) give that

$$0 \leq n!R_n < n!\frac{3}{(n+1)!} = \frac{3}{n+1} < 1.$$

Thus we have reached a contradiction. ☐

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QUESTION 4. (10 marks)

Suppose that

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Show that f is a differentiable at x for all $x \in \mathbb{R}$.

Proof. f is clearly differentiable at all $x \neq 0$. So it suffices to verify that $f'(0)$ exists. We have

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0.$$

This completes the proof. □

QUESTION 5. (10 marks)

Let g be differentiable on (a, b) and suppose that there is $m \in \mathbb{R}$ such that $|g'(x)| \leq m$ for all $x \in (a, b)$. Prove that g is uniformly continuous on (a, b).

Proof. Let $\varepsilon > 0$ be given. Set

$$\delta = \frac{\varepsilon}{m+1}.$$

Give $x, y \in (a, b)$ with $|x - y| < \delta$, by the mean value theorem, there is a t between x and y such that

$$|f(x) - f(y)| = |x - y||f'(t)| < \delta m < \varepsilon \frac{m}{m+1} < \varepsilon,$$

as desired. □

QUESTION 6. (25 marks)

Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous function and

$$\lim_{x \to \infty} \frac{\phi(x)}{x^n} = \lim_{x \to -\infty} \frac{\phi(n)}{x^n} = 0.$$

(i) Prove that if n is odd, then there is a number x such that

$$x^n + \phi(x) = 0.$$

Hint: If $\phi(x) \neq 0$, then one of the following should hold.

$$\lim_{x \to 0-} \frac{\phi(x)}{x^n} = -\infty \quad \lim_{x \to 0+} \frac{\phi(x)}{x^n} = -\infty$$

Hence there should be places where $\phi(x)/x^n < -1$ and the condition of the problem should imply that there should be places where $\phi(x)/x^n > -1/2$.

Proof. If $\phi(0) = 0$, then $x = 0$ satisfies the required identity. If $\phi(0) > 0$, then since n is odd,

$$\lim_{x \to 0-} \frac{\phi(x)}{x^n} = -\infty;$$

if $\phi(0) < 0$, then

$$\lim_{x \to 0+} \frac{\phi(x)}{x^n} = -\infty.$$

In the first case in which $\phi(0) > 0$, there exists $t < 0$ such that $\phi(t)/t^n < -1$. Moreover, since

$$\lim_{x \to -\infty} \frac{\phi(x)}{x^n} = 0,$$

there is $y < 0$ such that $\phi(y)/y^n > -1$. $\phi(x)/x$ is clearly continuous between t and y, so by the intermediate value theorem, there must be an x between the two quantities such that

$$\frac{\varphi(x)}{x^n} = -1 \text{ or equivalently } x^n + \phi(x) = 0.$$

The proof for the case $\phi(0) < 0$ is similar. □

(ii) Prove that if n is even, then there is a number y such that

$$y^n + \phi(y) \leq x^n + \phi(x).$$

Hint: First show that $\lim_{x\to\pm\infty} (x^n + \phi(x)) = +\infty$. Then it should be possible to find a large number $N > 1$ such that if $x \notin [-N, N]$, then

$$x^n + \phi(x) > \max_{x\in[-1,1]} (x^n + \phi(x)).$$

Now a theorem from the lectures should help us deal with the values inside $[-N, N]$.

Proof. We have

$$\lim_{x\to\pm\infty} 1 + \frac{\phi(x)}{x^n} = 1$$

which implies that

$$\lim_{x\to\pm\infty} x^n + \phi(x) = \lim_{x\to\pm\infty} x^n = \infty$$

as n is even. From this we can infer that there is $N > 1$ such that if $x \notin [-N, N]$,

$$x^n + \phi(x) > \max_{x\in[-1,1]} (x^n + \phi(x)).$$

Moreover, since $[-N, N]$ is clearly a compact set, there is $y \in [-N, N]$ such that

$$y^n + \phi(y) = \min_{x\in[-N,N]} (x^n + \phi(x)).$$

Also, if $x \notin [-N, N]$, then

$$y^n + \phi(y) = \min_{x\in[-N,N]} (x^n + \phi(x)) \leq \max_{x\in[-1,1]} (x^n + \phi(x)) < x^n + \phi(x).$$

This completes the proof. □