Suggested solutions for MAS326(2010-2012)

Question 1

Solution. (i) The basic solution determined by the basis $\{x_1, x_2, x_3\}$ is

$$(b_1 - b_2 - b_3, -b_1 + 2b_2 + b_3, b_3, 0, 0).$$

The corresponding complementary dual solution is

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-T} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1 - c_2 \\ -c_1 + 2c_2 \\ -c_1 + c_2 + c_3 \end{pmatrix}.$$

(ii) The tableau corresponding to the basis $\{x_1, x_2, x_3\}$ is

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	0	0	-2	2	8
0	1	0	0	1	-1	3
0	0	1	0	-1	2	0
0	0	0	1	1	2	1

By the largest coefficient rule, x_4 is the entering variable.

By the smallest subscript rule, using the minimum ratio test

$$\min \left\{ \frac{3}{1}, -, \frac{1}{1} \right\} = \frac{3}{1},$$

x_1 is the leaving variable. □

Question 2

Note: It should be $b^T = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$.

Solution. The basis $\{x_1, x_2, x_5, x_8\}$ determines the basic solution

$$(x_1^*, x_2^*, x_5^*, x_8^*) = (2, 3, 8, -2)$$

by

$$x_8^* = 3 - (x_1^* + x_2^* + x_3^* + x_4^*) = 3 - (2 + 3 + 0 + 0) = -2.$$

Hence x_8 is the leaving variable.

The inverse matrix B^{-1} is updated by adding the row

$$\begin{bmatrix} -a_{41} & -a_{42} & -a_{45} \end{bmatrix} B^{-1} = \begin{bmatrix} -1 & -1 & 0 \end{bmatrix} B^{-1} = \begin{bmatrix} 0 & -3 & -2 \end{bmatrix}$$

followed by adding the column $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$, giving

$$B^{-1} \leftarrow \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 3 & 4 & 0 \\ 0 & -3 & -2 & 1 \end{bmatrix}.$$

The dual solution is $\vec{y} = B^{-T}\vec{c}_B = \begin{bmatrix} 0 & 5 & 3 & 0 \end{bmatrix}^T$. The coefficients $\bar{a}_{8j}$ can be computed by

$$\begin{bmatrix} \bar{a}_{83} & \bar{a}_{84} & \bar{a}_{86} & \bar{a}_{87} \end{bmatrix} = (x_8\text{-row of } B^{-1}) \times N = \begin{bmatrix} 0 & -3 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 & -3 & -2 \end{bmatrix}.$$

The minimum ratio test

$$\min\left\{\frac{-1}{-2}, \_\_, \frac{-5}{-3}, \frac{-3}{-2}\right\} = \frac{-1}{-2} = \frac{1}{2}$$

shows that x_3 is the entering variable.

We now proceed to update the basic solution. Observe that

$$\vec{d} = B^{-1}\vec{a}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \end{bmatrix}.$$

The updated basic solution $\vec{x}^*$ has

$$x_3^* \leftarrow \frac{x_8^*}{d_8} = \frac{-2}{-2} = 1, \quad x_1^* \leftarrow x_1^* - d_1 \times \frac{x_8^*}{d_8} = 1,$$
$$x_2^* \leftarrow x_2^* - d_2 \times \frac{x_8^*}{d_8} = 1, \quad x_5^* \leftarrow x_5^* - d_5 \times \frac{x_8^*}{d_8} = 5.$$

Therefore, the basic solution x^* with $(x_1^*, x_2^*, x_3^*, x_5^*) = (1, 1, 1, 5)$ determined by the updated basis $\{x_1, x_2, x_3, x_5\}$ is optimal. □

Question 3

Solution. (a) Let J_B denote the basis determined by the spanning tree $\{AB, AC, CD, EA\}$. It is easy to see that, corresponding to basic variables $x_{AB}, x_{AC}, x_{CD}, x_{EA}$ and nonbasic variables x_{BE}, x_{BC}, x_{DE}, respectively,

$$c_B = (-1, -a, -7, -1), \quad c_N = (-1, -9, -1),$$

$$B = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

To ensure this basis is optimal, it should be

$$c_B^T B^{-1} N - c_N \geq 0.$$

Thus, we conclude that $0 \leq a \leq 10$.

(b) By the largest coefficient rule, t_{CD} is the entering variable, and x_{AC} is the leaving variable. □

Question 4

Solution. (a) Using the Northwest Corner rule, we get a feasible solution

$$(x_{11}, \dots, x_{34}) = (20, 0, 0, 0, 10, 20, 0, 0, 0, 20, 10, 30).$$

(b) For basic variable x_{14}, $u_1 + v_4 = 2$ with $u_1 = 0$ gives $v_4 = 2$.

For basic variable x_{24}, $u_2 + v_4 = c$ with $v_4 = 2$ gives $u_2 = c - 2$.

For basic variable x_{21}, $u_2 + v_1 = 3$ with $u_2 = c - 2$ gives $v_1 = 5 - c$.

For basic variable x_{31}, $u_3 + v_1 = 3$ with $v_1 = 5 - c$ gives $u_3 = c - 2$.

For basic variable x_{32}, $u_3 + v_2 = 2$ with $u_3 = c - 2$ gives $v_2 = 4 - c$.

For basic variable x_{33}, $u_3 + v_3 = c$ with $u_3 = c - 2$ gives $v_3 = 2$.

We now compute the reduced costs for the nonbasic variables using the formula $\bar{c}_{ij} = u_i + v_j - c_{ij}$. We have

$$\bar{c}_{11} = 2 - c, \ \bar{c}_{12} = 4 - 2c, \ \bar{c}_{13} = -2,$$
$$\bar{c}_{22} = -3, \ \bar{c}_{23} = c - 4, \ \bar{c}_{34} = c - 6.$$

If all the above $\bar{c}_{ij}$ are nonpositive, then the given basis is optimal. Thus, $c \geq 6$. □

Question 5

Solution. (i) Denote $f(x, y) := -e^{-x} - e^{-2y}$ and $g(x, y) := x + y$. The function $f(x, y)$ is concave since

$$\nabla^2 f(x, y) = \begin{bmatrix} -e^{-x} & 0 \\ 0 & -4e^{-2y} \end{bmatrix} \preceq 0$$

for any $(x, y) \in \mathbb{R}^2$. Meanwhile, the function $g(x, y)$ is convex since

$$g(\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2)) = \lambda g(x_1, y_1) + (1 - \lambda) g(x_2, y_2)$$

for all $(x_1, y_1), (x_2, y_2)$ and all $\lambda \in (0, 1)$. Thus this nonlinear program is a convex program.

(ii) The Karush-Kuhn-Tucker conditions are

$$\begin{array}{ll} (1)(a)e^{-x} - \lambda \leq 0, & (b)2e^{-2y} - \lambda \leq 0, \\ (2)(a)x(e^{-x} - \lambda) = 0, & (b)y(2e^{-2y} - \lambda) = 0, \\ (3)x + y \leq 1, & \\ (4)\lambda(1 - x - y) = 0, & \\ (5)x, y \geq 0, & \\ (6)\lambda \geq 0. & \end{array}$$

We consider for cases.

Case 1, $x, y = 0$:

In this case, condition (4) implies that $\lambda = 0$. This reduces condition (1a) to $1 \leq 0$, which is impossible. Therefore, there are no solutions in this case.

Case 2, $x > 0, y = 0$:

In this case, condition (2a) implies that $\lambda = e^{-x} > 0$. Condition (4) then implies that $x = 1$. Thus, $\lambda = e^{-1}$, which is in contradiction with condition (1b). Therefore, there are no solutions in this case.

Case 2, $x = 0, y > 0$:

In this case, condition (2b) implies that $\lambda = 2e^{-2y} > 0$. Condition (4) then implies that $y = 1$. Thus, $\lambda = 2e^{-2}$, which is in contradiction with condition (1a). Therefore, there are no solutions in this case.

Case 2, $x > 0, y > 0$:

In this case, condition (2) implies that $e^{-x} = \lambda = 2e^{-2y} > 0$, which results in $-x + 2y = ln2$. Condition (4) then implies that $x + y = 1$. Thus, $x = \frac{2-ln2}{3}, y = \frac{1+ln2}{3}$, and $\lambda = \sqrt[3]{2}e^{-\frac{2}{3}}$, which satisfies all KarushKuhn-Tucker conditions, and conclude that this is a solution to the Karush-Kuhn-Tucker conditions. □

Question 6

Solution. Suppose x^* is an n-vector and the number of inequalities is m. For a fixed n-vector c, by Theorem 4.1.5, there is a unique complementary dual solution y such that

$$y = B^{-T} c_B, \tag{1}$$

where B is the submatrix of coefficients of basic variables in the augmented form determined by the basis J_B, and c_B is the vector coefficients of the basic variables in the objective function.

As shown in the lecture, the reduced cost of x_j^* is

$$c_j - y_1 a_{1j} - \cdots - y_m a_{mj}, \quad j = 1, \ldots, n.$$

It is known that if a feasible basis has negative reduced costs for all nonbasic variables, then the BFS determined by the basis is the only optimal solution. Thus, if c is chosen to satisfy

$$c_j = y_1 a_{1j} + \cdots + y_m a_{mj}, \quad x_j^* \in J_B,$$
$$c_j < y_1 a_{1j} + \cdots + y_m a_{mj}, \quad j \in \{1, \ldots, n\} \text{ but } x_j^* \notin J_B,$$

where the above inequalities are in terms of $c_1, \ldots, c_n$ by (1), then the linear program has the unique optimal solution x^* □

Question 6

Proof. Suppose x^* is an n-vector and the number of inequalities is m. By Complementary Slackness Theorem, the dual of the linear program has a feasible solution y^* such that

$$(y_1^* a_{1j} + \cdots + y_m^* a_{mj} - c_j)x_j^* = 0 \text{ for } j = 1, \ldots, n, \tag{2}$$
$$(b_i - a_{i1}x_1^* - \cdots - a_{in}x_n^*)y_i^* = 0 \text{ for } i = 1, \ldots, m. \tag{3}$$

Since $Ax^* < b$, $y_i^* = 0$ for $i = 1, \ldots, m$ by (3). Thus, $c_j x_j^* = 0$ for $j = 1, \ldots, n$ by (2), whence $c^T x^* = 0$. □