NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER I EXAMINATION 2010-2011

MA1506 Mathematics II

Nov/Dec 2010 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **FIVE (5)** printed pages.
2. Answer **ALL** questions in this paper. Marks for each question are indicated at the beginning of the question.
3. Candidates may use non-graphing, non-programmable calculators. However, they should lay out systematically the various steps in the calculations.
4. One A4 handwritten double-sided helpsheet is allowed.

Answer all the questions.

Marks for each question are indicated at the beginning of the question.

Question 1 [20 marks]

(a) Let $y(x)$ be the solution of the initial value problem

$$\frac{dy}{dx} = -y^2 \sin x, \qquad y(0) = 1.$$

Find the value of $y(2\pi)$.

(b) Consider the system of differential equations

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix},$$

where A is a 2×2 matrix. Given that A has an eigenvalue $4+2i$ whose corresponding eigenvector is $\begin{bmatrix} 4 \\ -1+i \end{bmatrix}$. Find the general solution of the system.

(c) (i) Let z be a differentiable function of x. Prove that

$$\frac{d}{dx}(z^2) = 2z\frac{dz}{dx}.$$

(ii) Let $y(x)$ be the solution of the differential equation

$$2x \sin y \cos y \frac{dy}{dx} = x^2 + \sin^2 y, \qquad y(1) = \frac{\pi}{2}.$$

If $0 \leq y(x) \leq \frac{\pi}{2}$ for $0 \leq x \leq 1$, find the value of $y(\frac{1}{2})$.

(d) Find a particular solution of the differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = \cos(e^x).$$

Question 2 [20 marks]

(a) A mass-spring system (with unit mass) satisfies the differential equation

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0,$$

$$x(0) = A, \qquad \frac{dx}{dt}(0) = B.$$

Suppose that

$$x(t) = 4\cos(4t) - 3\sin(4t)$$

is the solution of the differential equation.

(i) Find the values of A and B.

(ii) Find the values of b and k.

(b) A cantilevered beam of length $2L$, made up of extremely strong and light material, is horizontal at the end where it is attached to a wall and carries a load of F Newtons at $x = L$, where x is the distance from the fixed end of the beam. Assuming that the weight of the beam is negligible so that the deflection $y(x)$ of the beam is given by

$$\frac{d^4y}{dx^4} = -\frac{F\delta(x-L)}{EI},$$

where $\delta(x)$ is the dirac delta function at $x = 0$ and E, I are constants. Find the value of $\frac{d^2y}{dx^2}$ at $x = \frac{3L}{2}$. Justify your answer.

[You may assume that $\frac{d^2y}{dx^2}(2L) = \frac{d^3y}{dx^3}(2L) = 0$.]

(c) The population $y(t)$ of a certain species of fish in a lake can be modeled by the following differential equation

$$\frac{dy}{dt} = y(y-1)(1-\frac{y}{k}),$$

where $k > 2$. If the initial population of the fish is $\frac{k}{2}$, find the fish population in the long run. Express your answer in terms of k.

(d) The current $i(t)$ in a LCR circuit satisfies the differential equation

$$\frac{di(t)}{dt} + 110i(t) + 1000q(t) = 90 - 90u(t-1),$$

where $q(t)$ is the charge on the capacitor and $u(t)$ is the unit step function at $t = 0$. If $i(0) = 0$ and $q(0) = 0$, find the Laplace transform of the current $i(t)$ in the circuit.

[You may assume that $\frac{dq(t)}{dt} = i(t)$.]

Question 3 [20 marks]

(a) Let $y(x)$ be the solution of the initial value problem

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{\cos x}{x^2}, \qquad y(\pi) = 0.$$

Find the value of $y(2\pi)$.

(b) Consider the system of differential equations

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -8 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

(i) Find the equilibrium point.

(ii) Classify the equilibrium point as one of the six types discussed in the lecture.

(iii) Sketch a phase plane diagram for the system. Your sketch should have enough solution curves to show what happens in every part of the phase plane diagram. Use arrows to indicate the direction of motion for each solution curve.

(c) A tank has a volume of 30 litres. Initially, it has 10 litres of salt solution and the concentration of salt in the solution is 2 kg per litre. A salt solution of concentration 0.5 kg per litre is being poured into the tank at a constant rate of 2 litres per minute. The well-mixed solution is constantly being pumped out of the tank at a rate of 1 litre per minute. Find the amount of salt (in kg) in the tank at the time when the tank becomes full, that is, it has 30 litres of salt solution.

(d) Cats like to eat rats. Let C_k and R_k be the number of cats and rats in a given area at the k^{th} month of observation. Their populations can be modeled by the following equation

$$C_{k+1} = 0.5C_k + 0.4R_k$$
$$R_{k+1} = -0.1C_k + 1.1R_k.$$

(i) Suppose that there are 2000 rats and 250 cats when $k = 0$. Sketch the solution curve in a phase plane diagram.

(ii) If the cats and rats survive in the long run, find the ratio of cats to rats after a long time.

Question 4 [20 marks]

(a) Find the value (or values) of t such that the following vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ t \\ 1 \end{bmatrix}, \begin{bmatrix} t \\ 1 \\ 1 \end{bmatrix}$$

are linearly dependent in $\mathbb{R}^3$. Justify your answer.

(b) Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find e^A. Express your answer as a single matrix.

(c) The square root of a matrix A is a matrix R such that $RR = A$. Determine whether the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has any square root. Justify your answer.

(d) A company has two interacting branches, A and B. For each \$1 worth of product A produces, it requires \$0.30 worth of its own product and \$0.50 worth of the product B produces. For every \$1 worth of product B produces, it requires \$0.30 worth of its own product and \$0.40 worth of the product A produces.

Suppose the company has a daily external demand of \$150 for A's product and \$100 for B's product. Find a production schedule to meet the external demand.

END OF PAPER